ON A MODEL FOR RANK ANALYSIS IN PAIRED COMPARISONS

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SUMMARY

A method of testing the appropriateness of the mathematical model for analysis of experiments involving ranking in paired comparisons has been developed. The likelihood ratio test is proposed for testing the goodness of fit. For large n this statistic turns out to be χ^2 test. The mathematical model is also suitable for analysing the experiments conducted in different groups. A procedure has been obtained to test the goodness of fit in such cases also. When this test is converted to observed and expected frequencies, this would be usualy χ^2 test of goodness of fit. Formulae for the variances and covariances of estimates of treatment ratings $\pi_1, \ldots\ldots \pi_t$ have been obtained, For testing the suitability of the model a numerical example from taste-testing experiment is given. The estimate of variances and covariances of the estimator of the treatment ratings have been worked out.

Statistical methodology applied to numerical data is appropriate and valid only when the data conform the assumptions and requirements needed in the analysis. The mathematical model for paired comparisons (Gupta and Rai [7]) was postulated in such a way that the estimates were mathematically workable and the model was easy to apply and interpret. In the present paper we propose to develop procedure for testing the appropriateness of the model and also to investigate the reliability of the estimaters used in paired comparisons.

2. Review of the Model and Tests of Hypothesis

In paired comparisons, the existence of non-negative parameters $\pi_1, \pi_2, .., \pi_t$ associated with t treatments $T_1, T_2, \ldots, T_t$ is postulated such that $\sum_{i=1}^{t} \pi_i = 1$. The behaviour of the parameters is further defined with probability statement that

$$P(T_i > T_j) = \pi_i^2 / (\pi_i^2 + \pi_j^2) \quad \ldots(1)$$

The model at (1) is quicker and sharper than Bradley-Terry model [1] of paired comparisons for detecting the quality of better treatment. For example, if T_i is superior to T_j, then π_i will be greater than π_j and the probability that T_i is preferred over T_j is higher in model at (1) than the corresponding probability obtained by using Bradley-Terry model *i.e.* $P(T_i > T_j) = \pi_i / (\pi_i + \pi_j)$.

In case of equality of treatment effects between T_i and T_j, the preference probability of one over the other is the same in both the models. In this way the present model is capable of detecting even a small difference in the treatment effects with higher probability. In the comparison of treatments, observations are limited to rankings of items in pairs. The rank one is assigned to the treatment of a pair which is judged superior on the basis of the test attribute and two to the other treatment. Tied ranks are not permitted in the model and this makes r_{ijk} to take a value either one or two with $r_{ijk} + r_{jik} = 3$, where r_{ijk} is the rank of T_i when it is compared with T_j in the K-th comparison. If the design is repeated n times, the likelihood function is given by

$$L = \frac{\Pi_i \pi_i^{4n(t-1) - 2 \sum\limits_{i \neq j = 1}^{t} \sum\limits_{k=1}^{n} r_{ijk}}}{\prod\limits_{i<j} (\pi_i^2 + \pi_j^2)^n} \quad \ldots(2)$$

The maximum likelihood estimator of π_i is denoted by p_i. Procedures for obtaining these estimators are outlined by Gupta and Rai [7]. Here we shall require to refer to (2) for developing the theory for testing the appropriateness of the model. Following the arguments and methodology given by Dykstra [6], we can modify the model for the cases of unequal number of repetitions on pairs. This is specially required to judge the equality of a new treatment as compared to the existing treatments.

3. Test for the Appropriateness of the Model

Here we limit ourselves to a test of the goodness of fit for the paired model for t treatments and an homogeneous repetitions of the possible pairwise treatment comparisons.

We define a parameter $\bar{\pi}_{ij}$, probability that T_i is ranked above T_j where T_i and T_j are compared. The complementary probability is given by $\bar{\pi}_{ji}=1-\bar{\pi}_{ij}$. If t treatments are considered in an experiment of this type, $\binom{t}{2}$ parameters are to be estimated. Also we have

$$\bar{p}_{ij}=f_{ij}/n \qquad \ldots(3)$$

where $\bar{p}_{ij}$ is the estimator of $\bar{\pi}_{ij}$; f_{ij} is the number of times T_i is ranked above T_j and n is the number of times such a comparison is made. The likelihood function may be written as

$$\bar{L}(\bar{\pi}_{ij})=\prod_{i,j} \bar{\pi}_{ij}^{f_{ij}}\, \bar{\pi}_{ji}^{f_{ij}} \qquad \ldots(4)$$

where $\qquad f_{ij}+f_{ji}=n \qquad$ and $\qquad \bar{\pi}_{ij}+\bar{\pi}_{ji}=1$

We are interested in the following test :

Test of fit I

$$H_0 : \bar{\pi}_{ij}=\pi_i^2/(\pi_i^2+\pi_i^2);\ i\neq j;\ i, j=1, 2,\ldots, t$$

$$H_1 : \bar{\pi}_{ij}\neq \pi_i^2/(\pi_i^2+\pi_i^2);\ \text{for some } i \text{ and } j$$

The likelihood ratio statistic depends on $\bar{L}(\bar{p}_{ij}/H_0)$ and $\bar{L}(\bar{p}_{ij}/H_1)$ where $\bar{L}$ is defined by (4) above.

Also we note that

$$\bar{L}(\bar{p}_{ij}/H_0)=L(p_i) \text{ where } L \text{ is given by}$$

$$L(\pi_i)=\prod_i \pi_i^{4n(t-1)-2\sum\limits_{i\neq j}^{t}\sum\limits_{k=1}^{n} r_{ijk}} \prod_{i<j}(\pi_i^2+\pi_j^2)^{-n} \qquad \ldots(5)$$

Also we have

$$\log\bar{L}(\bar{P}_{ij}/H_0)=\sum_i \left\{ 4n(t-1)-\sum_{i\neq j}^{t}\sum_{k=1}^{n} r_{ijk}\right\} \log p_i$$

$$-n\sum_{i<j} \log (p_i^2+p_j^2)=B \log 10 \qquad \ldots(6)$$

where $\qquad B=n\sum\limits_{i<j} \log(p_i^2+p_j^2)-\sum a_i \log p_i$

and $$a_i=4n(t-1)-2\sum_j\sum_k r_{ijk}$$

and $$\log \overline{L}(\overline{p}_{ij}/H_1)=\sum_{i\neq j} f_{ij}\log f_{ij}-n\binom{t}{2}\log n \qquad \ldots(7)$$

The goodness of fit is tested by the statistic

$$-2\log\lambda_1=2\left[\sum_{i\neq j} f_{ij}\log f_{ij}-n\binom{t}{2}\log n+B\log 10\right] \qquad \ldots(8)$$

which has the χ^2 distribution with $\binom{t}{2}-t+1$ degree of freedom for large values of n.

The computations become fairly easy after forming the following two way table.

TABLE 1

Table for the Test of Goodness of Fit

	T_1	T_2	T_3	r_i
T_1	—	f_{12}	f_{13}	$r_1=8n-f_{12}-f_{13}$
T_2	f_{21}	—	f_{23}	$r_2=8n-f_{21}-f_{23}$
T_3	f_{31}	f_{32}	—	$r_3=8n-f_{31}-f_{32}$

In case the repetitions of the paired comparison experiments are grouped into g groups, the u-th group containing n_u repetitions and $\sum_{u=1}^{g} n_u=n$; separate sets of parameters π_{iju} and π_{iu} are assumed to exist within each group. For each of the g groups, we calculate the statistics f_{iju} and $\sum_{i=1}^{t} r_{iu}$. The within group test of goodness of fit can be made as described above, we will denote the test statistic by $-2\log\lambda_{lu}$.

The test procedure is as follows :

Test of Fit II

$$H_0:\overline{\pi}_{iju}=\pi_{iu}^2/(\pi_{iu}^2+\pi_{ju}^2),\quad i\neq j,\ i,j=1,2,\ldots,t.$$

$$H_1:\overline{\pi}_{iju}\neq\pi_{iu}^2/(\pi_{iu}^2+\pi_{ju}^2)$$

for some i, j and u,

The likelihood ratio statistic is

$$-2 \log \lambda_{11} = -2 \sum_{u=1}^{g} \log \lambda_{1u}\,, \qquad \ldots(9)$$

which follows a χ^2 distribution with $\{g\binom{t}{2}-t+1\}$ degree of freedom for large values of n_u.

4. The Test of Goodness of Fit Associated with Expected Frequencies

The relative frequencies are related to the sums of ranks of treatments and are given by

$$2 \sum_{j \neq i} \sum_{k} r_{ijk} = 4n\,(t-1) - \sum_{j} f_{ij} \qquad \ldots(10)$$

We also write $\sum_{i} r_i$ as an abbreviation for the left hand member and refer to it as the total sum of ranks for T_i.

The expected cell frequencies are denoted by f'_{ij}, and are related to the estimator $p_1, p_2, \ldots, p_t$ through

$$f'_{ij} = np_i^2 \,/(p_i^2 + p_j^2) \qquad \ldots(11)$$

Thus using (8) and (6) we get

$$B = -\sum_{i \neq j} f_{ij} \log\,(f'_{ij}\,/n)$$

and

$$-2 \log \lambda_1 = 2 \sum_{i \neq j} f_{ij} \log\,(f_{ij}/f'_{ij}\,)$$

we may take

$$f_{ij}/f'_{ij} = 1 + \epsilon_{ij}$$

where ϵ_{ij} may be either positive or negative small numbers such that

$$\sum_{i \neq j} f'_{ij} \times \epsilon_{ij} \ldots = 0$$

Now

$$-2\log \lambda_1 = 2 \sum_{i \neq j} f'_{ij} \; (i + \epsilon_{ij}) \log (i + \epsilon_{ij})$$

Using power series expansions for the logrithms and stopping with the second term we get

$$-2 \log \lambda_1 \approx \sum_{i \neq i} f'_{ij} \, \epsilon^2_{ij} - \sum_{i \neq i} f'_{ij} \, \epsilon^3_{ij} \, . \qquad \ldots(12)$$

Finally we obtain,

$$-2 \log \lambda_1 \approx \sum_{i \neq j} \frac{(f_{ij} - f'_{ij})^2}{f'_{ij}}$$

After neglecting the terms involving ϵ^3_{ij} and higher power of ϵ_{ij}. This is the usual form of the test.

For group of repetitions we can write it as

$$-2\log \lambda_{11} = \sum_{u=1}^{g} \sum_{i \neq j} \frac{(f_{iju} - f'_{iju})^2}{f'_{iju}} \qquad \ldots(13)$$

For practical purposes the two methods of computing χ^2 for tests of goodness of fit or of the appropriateness of the models will usually be equivalent. When the values of B are available, the χ^2 can be readily calculated.

If the observed frequencies are small, we can not use (12) or (13). Further χ^2 test developed is only approximate and a large number of repetitions is needed.

5. Asymtotic Distribution of the Estimator of π_i

Let us define X_i as the number of times treatment T_i obtain a ranking of unity in a repetitions of paired comparisons. The likelihood function in terms of X_i is given by

$$f(X, \pi)=\prod_{i} \pi_i^{X_i} \prod_{i<j} (\pi_i^2 + \pi_j^2)^{-1}$$

where X and π represent vectors $(X_1, \ldots, X_t)$ and $(\pi_1, \pi_2, \ldots, \pi_t)$ respectively. If $X_{i(k)}$ is the observation on X_i in the k-th of n repetitions, then we have

$$\sum_{k=1}^{n} X_{i(k)}=a_i, \ (i=1, 2, \ldots, t)$$

where a_i is as usual given by

$$a_i=4\, n(t-1)-2\sum_{j\neq i} r_{ijk}$$

and the likelihood function may be written as

$$\prod_{k=1}^{n} f(X_{(k)}),$$

We define

$$d_{i1}=\frac{1}{\pi_i^2}\sum_{j\neq i} \pi_j^2\, (\pi_i^2 + \pi_j^2)^{-2} \qquad (i=1, 2, \ldots, t) \quad \ldots(14)$$

$$d_{ij}=-(\pi_i^2 + \pi_j^2)^{-2} \qquad (i\neq j,\ i, j=1, \ldots, t)$$

We shall require the means, variances and covariances of $X_1, X_2, \ldots, X_t$. We define X_{ij} such that, it has value unity if treatment T_i ranks above treatment T_j and zero otherwise.

Then

$$X_i=\sum_{j\neq 1} X_{ij}$$

Thus x_{ij} is a Bernoulli variate with expectation $\pi_i^2\, (\pi_i^2 + \pi_i^2)^{-1}$ and variance $\pi_i^2 \pi_j^2\, (\pi_i^2 + \pi_j^2)^{-2}$. The variates X_{ij} making up the sum X_i are independent in probability and it follows that

$$E(X_i)=\pi_i^2 \sum_{j\neq i} (\pi_i^2 + \pi_j^2)^{-1} \quad \ldots(i=1, 2, \ldots, t) \qquad \ldots(15)$$

$$V(X_i)=\pi_i^2 \sum_{j\neq i} \pi_j^2 (\pi_i^2+\pi_j^2)^{-2} \quad (i=1, 2, \cdots, t) \qquad \ldots(16)$$

$$=\pi_i^4 d_{ii}$$

$$\text{Cov}(X_i,X_j)=-\pi_i^2 \pi_j^2 (\pi_i^2+\pi_j^2)^{-2}$$

$$=\pi_i^2 \pi_j^2 d_{ij} \quad (i\neq j,\ i, j=1, 2, \ldots, t) \qquad \ldots(17)$$

The parameters $\pi_i, \ldots, \pi_t$ are subject to the restriction $\sum_i \pi_i=1$ and are not independent. Thus we may regard $p_1, p_2, \ldots, p_{t-1}$ maximum likelihood estimators of independent parameters $\pi_i, \pi_2, \ldots, \pi_{t-1}$ taking $\pi_t=1-\sum_{i=1}^{t-1} \pi_i$. Then $\sqrt{n}(p_1-\pi_1), \sqrt{n}(p_2-\pi_2), \ldots, \sqrt{n}(p_{t-1}-\pi_{t-1})$ have joint limiting normal distribution with zero means subject to the verification of certain regularity conditions (Cramer [5] and Chanda [4]). $(d_{ij})^{-1}$ denotes the dispersion matrix of the joint limiting normal distribution of $(t-1)$ estimation with d'_{ij} as given below : we note that

$$\frac{\partial}{\partial \pi_i}(\log f)=X_i/\pi_i-X_t/\pi_t-2\pi_j \sum_{j\neq i} (\pi_i^2+\pi_i^2)^{-1}$$

$$+2\pi_t \sum_{j\neq i} (\pi_i^2+\pi_j^2)^{-1} \quad (i=1, 2, \ldots, t)$$

and using $E(X_i)=\pi_i^2 \sum_{j\neq i} (\pi_i^2+\pi_j^2)^{-1}$ we get

$$E\left(\frac{\partial}{\partial \pi_i}\log f\right)\left(\frac{\partial}{\partial \pi_j}\log f\right)=\text{Cov}\left(\frac{X_i}{\pi_i}-\frac{X_t}{\pi_t}\right)$$

$$\left(\frac{X_j}{\pi_j}-\frac{X_t}{\pi_t}\right)=d'_{ii};\ (i, j=1, 2, \ldots t)$$

there by defining d'_{ij}

Also from (14) and (15) we get $d'_{ij}=d_{ij}-d_{it}-d_{jt}+d_{tt}$ ...(18)

$$\text{Since } d'_{ij}=\text{Cov}\left[\frac{X_iX_j}{\pi_i\pi_j}-\frac{X_iX_t}{\pi_i\pi_t}-\frac{X_tX_j}{\pi_t\pi_j}+\frac{X_t^2}{\pi_t^2}\right]$$

The matrix (d'_{ij}) is non-negative definite because it is a dispersion matrix and hence $\frac{\partial}{\partial \pi_1}\log f, \ldots, \frac{\partial}{\partial \pi_{t-1}}\log f$ are free of linear restrictions.

Thus we conclude that $\sqrt{n}(p_1-\pi_1), \ldots, \sqrt{n}(p_{t-1}-\pi_{t-1})$ have the multivariate normal distributions with zero means and dispersions matrix $(d'_{ij})^{-1}$. This result holds true for large sample.

6. Variances and Covariances of Estimators

We note that the $t \times t$ matrix (d_{ij}) is singular in view of definition (14) and

$$\pi_i^2 d_{ij} + \pi_j^2 d_{ij} = 0$$

If the elements of the last row and then of the last column are subtracted from corresponding elements of the remaining rows and columns respectively, we obtain

$$| d_{ij} | = \begin{vmatrix} (d'_{ij}) & (d_{it}-d_{tt})' \\ (d_{jt}-d_{tt}) & d_{tt} \end{vmatrix} = 0$$

where $(d_{it}-d_{tt})'$ and $(d_{jt}-d_{tt})$ are respectively column and row vectors of $(t-1)$ elements. Thus

$$| d'_{ij} | = \begin{vmatrix} (d'_{ii}) & (d_{it}-d_{tt})' \\ (d_{jt}-d_{tt}) & 1+d_{tt} \end{vmatrix} \qquad \ldots(19)$$

Now we reverse the porcess and add the elements of the last row and then the last column to corresponding elements of the remaining rows and columns in (19) above, we get

$$| d'_{ij} | = | d_{ij}+1 | = \begin{vmatrix} (d_{ij}+1) & (1)' \\ (0) & (1) \end{vmatrix} = -\begin{vmatrix} (d_{ij}) & (1)' \\ (1) & (0) \end{vmatrix} \qquad \ldots(20)$$

Since $\qquad | d_{ij} | = 0$

where (1) and (1)′ are respectively row and column vectors of t unit elements.

Similarly we can show that the cofactor of d_{ij} in the extreme right side of (18) is equal to the cofactor of d'_{ij} in $| d'_{ij} |$.

Thus if σ_{ij} is the covariance of $\sqrt{n}(p_i-\pi_i)$ and $\sqrt{n}(p_j-\pi_j)$, $(i, j=1, 2, \ldots, t)$

$$\sigma_{ij} = \frac{\text{cofactor of } d_{ij} \text{ in } \begin{vmatrix} (d_{ij}) & (1)' \\ (1) & 0 \end{vmatrix}}{\begin{vmatrix} (d_{ij}) & (1)' \\ (1) & 0 \end{vmatrix}} \qquad \ldots(21)$$

Since the formulation of the model for paired comparison is symmetrical in the parameters $(\pi_1, \pi_2, \ldots, \pi_t)$ and the estimator $(p_1, p_2, \ldots, p_t)$ applies to all variances and covariances with $(i, j=1, 2, \ldots, t)$ on the basis of symmetry.

The variances and covariances $\sigma_{ij}, (i, j=1, \ldots, t)$ are simply the elements of the $t-$square principal minor of the inverse of the matrix $\begin{vmatrix} (d_{ij}) & (I)' \\ (I) & 0 \end{vmatrix}$

When t is small, the inverse can be obtained by elementary methods when $\pi_1, \ldots, \pi_t$ are replaced by estimates or they are specified. However, since $|d_{ij}|=0$, the usual Doo-little methods breakdown. Then for larger values of t, it is desirable to invert (d'_{ij}) specified through (18) using a Doolittle method. The remaining variance and covariance of $\sqrt{n}(p_t-\pi_t)$ are obtained from

$$\sqrt{n}(p_t-\pi_t)=-\sum_{i=1}^{t-1} \sqrt{n}\,(p_i-\pi_i)$$

so that $$\sigma_{tt}=\sum_{i=1}^{t-1}\sum_{j=1}^{t-1} \sigma_{ij} \text{ and } \sigma_{it}=-\sum_{j=1}^{t-1} \sigma_{ij}\ (i=1, 2, \ldots, t-1)$$

Since $\sqrt{n}(p_t-\pi_t)$ is a linear function of $\sqrt{n}(p_1-\pi_1), \ldots, \sqrt{n}\,(p_{t-1}-\pi_{t-1})$ we may state that $\sqrt{n}(p_1-\pi_1), \ldots, \sqrt{n}(p_t-\pi_t)$ have for large values of n, the singular multivariate normal distribution of $(t-1)$ dimensions in a space of t dimensions with zero means and dispersion matrix (σ_{ij}) defined above.

In general we may take $p_1, \ldots, p_t$ jointly normally distributed with means $\pi_1, \ldots, \pi_t$ and dispersions matrix $(\sigma_{ij})/n$ for large samples and any linear function $\Sigma\, bipi$ have normal distribution with mean $\sum_i b_i\pi_i$ and variance $\sum_{i<j} b_ib_j\, \sigma_{ij}/n$ for large samples.

Usually, estimated variances and covariances are required. We may define $d_{ij}(i, j=1, 2, \ldots, t)$ to be the same function of $p_1, p_2, \ldots, p_t$ as d_{ij} are of $\pi_1, \ldots, \pi_t$ as defined in (12) since maximum likelihood estimates are consistent. Thus $(\hat{\sigma}_{ij})$ is the dispersion matrix which is the same function of $\hat{d}_{ij}$ as (σ_{ij}) is of d_{ij} in (19).

7. Illustrative Example

The procedures developed in this paper will be demonstrated by the numerical example taken from a taste-testing experiment conducted by the author at IASRI. Three popular brands of tinned mango juice were compared for their taste quality in paired comparisons design. The mango juices were coded as follows :

T_1 = Sun-ship Mango

T_2 = Mohan Mango Juice

T_3 = Noga Mango Juice.

The juices were presented in pairs to the judges for tasting one after the other in random order and they were asked to record their preferences interms of ranks. The paired comparison design was repeated ten times. The pooled sum of ranks for two julges are 32,28 and 39 respectively for T_1,T_2 and T_3. The experiment can be analysed by applying Bradley-Terry model [1]. But here we shall demonstrate the use of the model developed in the paper.

The estimated value of p_1,p_2 and p_3 may be obtained by using the table given by Gupta and Rai [7]. The value of p_1,p_2, and p_3 are respectively 0.29, 0.38 and 0 33. This result is non-significant at any realistic level of significance and is not indicative of any real difference in the taste quality of mango juice. From Bradley-Terry model we also reach the same conclusion.

The values of $\hat{d}_{ij}$ are obtained by substituting values of p_i for π_i in (14).

$\hat{d}_{11} = 67.6479$ $\qquad$ $\hat{d}_{22} = 22.9087$

$\hat{d}_{12} = -19.1525$ $\qquad$ $\hat{d}_{23} = -15.5858$

$\hat{d}_{13} = -26.8463$ $\qquad$ $\hat{d}_{33} = 41.3990$

The estimate of the determinant in the denominator of (21) is

$$\begin{vmatrix} 67.6479 & -19.1525 & -26.8463 & 1 \\ -19.1525 & 22.9028 & -15.5858 & 1 \\ -26.8463 & -15.5858 & 41.3990 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$= -11354.9321$

Now from (21)

$$\hat{\sigma}_{11} = \frac{1}{11354.9321} \begin{vmatrix} 22.9087 & -15.5858 & 1 \\ -15.5858 & 41.3990 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$= 0.008408$

Similarly, complete set of variances and covariances is

$\hat{\sigma}_{11} = 0.008408$ $\hat{\sigma}_{22} = 0.01423$

$\hat{\sigma}_{12} = -0.005696$ $\hat{\sigma}_{23} = -0.008036$

$\hat{\sigma}_{13} = -0.002712$ $\hat{\sigma}_{33} = 0.01134$

The estimated variances and covariances of p_1, p_2, p_3 are obtained by dividing those given above by $n=10$. Consequently the standard errors of p_1, p_2, p_3 are obtained as

S.E. $(p_1) = 0.289$; S.E. $(p_2) = 0.038$, S.E. $(p_3) = 0.0337$

A check on the calculations is provided by calculating the variance of $\sqrt{n}\sum_i p_i$ in terms of variances and covariances of the elements of this sum which should obviously by zero.

For testing goodness of fit of the model, we give below the χ^2 computed for one judge in detail. The χ^2 computed has one degree of freedom and is calculated using both the forms (8) and (12), the latter value being shown in parentheses. The 5 percent level of χ^2 with 1 degree of freedom is 3.84 and the 1 percent level is 6.63.

Results for Judge I

	A	B	C	Σr_i
		7	5	28
A	—	(6.00)	(6.00)	
	3	—	6	
B	(4.00)	—	(5.00)	31
	5	4	—	
C	(4.00	(5.00)	—	31

$t=3,\ n=10,\ B=0.586,\ \chi_1^2=1.24\ (1.23)$

Significance level of treatment$=0.78$

Using test of fit in the form (8), we get,

$$-2 \log_e \lambda_1 = 2\ (2.3026)\ (7 \log 7 + 5 \log 5 + 3 \log 3 + 6 \log 6 + 5 \log 5 + 4 \log 4 + 8.856 - 30 \log 10) = 1.24$$

To use the second form (12), we require $PA=0.38$; $PB=0.31$: $PC=0.31$ from tables Gupta and Rai [7]. The expected cell frequencies are shown in parentheses in table above and they were calculated using the relationship (11).

The alternative method of computing χ^2 yields

$$\chi_1^2 = \frac{(7-6)^2}{6} + \frac{(5-6)^2}{6} + \frac{(3-4)^2}{4} + \frac{(6-5)^2}{5} + \frac{(5-4)^2}{4} + \frac{(4-5)^2}{5}$$

$$= 1.23$$

The two methods of computing χ^2 for test of fit I yield the values in close agreement and indicate good agreement of the model and the observations.

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